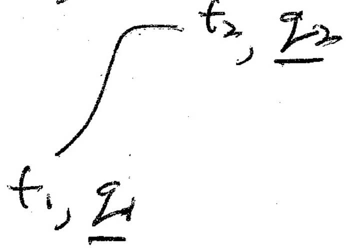
1.) Review of Lagrangian Mechanics

Physics 200A Mechanics Fall '10 2003

I.) Basic Lagrangian Dynamics

→ Principle of Least Action / Hamilton's Principle

If mechanical system;

- parametrized by generalized coordinates (need not correspond to usual coordinate system) $q_1 \ldots q_s$; generalized velocities $\dot{q}_1, \ldots \dot{q}_s$; t
- 

i.e. path end-points, times known
- system described by $L(\underline{q}, \underline{\dot{q}}, t)$ ↓ Lagrangian

then path $\underline{q}_1(t_1) \to \underline{q}_2(t_2)$ is one which minimizes

$$S = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t)$$

↓ Action

⇒ i.e. path selected by principle of least action, S

Significance:

- variational principle allows use of generalized coordinates — convenient to problem

  i.e. S minimal, parametrization independent

- $L = T - V \Rightarrow$ energy methods. ↳ any g.c.'s ok

Now, path $q(t)$ s/t.

$\delta S = 0$ (necessary for S minimal)

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right)$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q + \frac{\partial L}{\partial q} \delta q \right)$$

$$= \underbrace{\cancel{\frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2}}}_{0} + \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q$$

$\delta q = 0$ on end-points (fixed)

if $\delta S = \frac{\partial L}{\partial \dot{q}} \delta q$ (sets L.E.)

so $\delta S = 0$ for all δq if

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$

Lagrange's Eqns.

- Lagrange's eqns. determine trajectory $q(t)$
- generally pde's for $q_i(t)$
- observe adding $\frac{df}{dt}$ (i.e. total time derivative) leaves Lagrangian invariant
  $\to$ fctn. of gen. coords.

$$L \to L + df/dt$$

$$S' = \int dt\, L = \int dt\,(L + df/dt) = S + f\Big|_{t_2} - f\Big|_{t_1}$$

$\delta q = 0$

$\delta S' = \delta S \Rightarrow$ no change in trajectory (physics!).

Now, 2 obvious questions:

a) - what's L?

b) - is $q(t)$ a minimum of S? - later
(i.e. have proved soln. of Lagrange's Eqns. is extremum)

a) Structure of L

Action Extrema Invariant Under

$$L \to L + \frac{df}{dt} \qquad f = f(q,t)$$

($\underline{\text{not}}$ function of $\dot{q}$)

Now, action extrema $\Leftrightarrow$ $\delta S = 0$.

$$S = \int_{t_1}^{t_2} \left(L + \frac{df}{dt} \right) dt$$

also relevant to constr.

$$= S_0 + f \Big|_{t_1}^{t_2}$$

$$\delta S = \delta S_0 + \frac{\partial f}{\partial q} \delta q \Big|_{t_1}^{t_2}$$

but $\delta q(t_2) = \delta q(t_1) = 0$

$$\therefore \delta S = \delta S_0$$

Thus, extrema invariant under $L \to L + df/dt$.

symmetry $\Rightarrow$

for free particle (non-relativistic),

→ space-time homogeneity $\Rightarrow$ L cannot depend on $\underline{x}, t$; only on $\underline{v}$

→ space-time isotropy $\Rightarrow$ L depends on $v^2 = \underline{v} \cdot \underline{v}$, only (not $\underline{v}$) direction!

$\therefore$ $L = L(v^2)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}}\right) - \cancel{\frac{\partial L}{\partial \underline{r}}} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial v^2} 2\underline{v}\right) = 0 \Rightarrow \begin{cases} \underline{v} = \text{const} \\ \text{Law of inertia (Galileo/Newton)} \end{cases}$$

↳ $\frac{\partial L}{\partial v^2} = \frac{m}{2}$ (from Galilean Relativity + sees) → $\boxed{\text{establish } \partial L / \partial v^2 !}$ → 5

$\underline{v} = \text{const.}$

so to relate 2 frames with relative velocity $\underline{V}$

$$\begin{cases} \underline{r} = \underline{r}' + \underline{V} t \\ t = t' \end{cases}$$

Now $\rightarrow L = L(v^2)$

→ Principle of (Galilean) Relativity ⇒ For two frames related by infinitesimal Galilean boost, trajectories must be the same, total deriv.
↳ same action

$$\Rightarrow \quad L((\underline{v}+\delta \underline{v}))^2, \quad L(\underline{v}^2) \quad \text{differ by} \quad df/dt$$

$$L[(\underline{v}+d\underline{v})^2] - L(\underline{v}^2) \cong \cancel{L(\underline{v}^2)} + (2\underline{v}\cdot d\underline{v} + \cancel{d\underline{v}^2})\frac{\partial L}{\partial v^2} - \cancel{L(\underline{v}^2)}$$

$$= 2\underline{v}\cdot d\underline{v}\,\frac{\partial L}{\partial v^2} \equiv \Delta L$$

$\Delta L = \dfrac{df}{dt}$ $\underline{\underline{\text{iff}}}$ $\dfrac{\partial L}{\partial v^2}$ independent v (i.e. constant), so ΔL linear in $\underline{v}$

$$\Rightarrow \boxed{\frac{\partial L}{\partial v^2} = \text{const} = \frac{m}{2}} \; ; \quad \left\{ \begin{array}{l} m>0 \text{ for} \\ \underline{\text{minimum}} \text{ in } S \end{array} \right.$$

thus, for free particle,

$$L = \frac{1}{2} m v^2$$

↳ kinetic energy

For system of free particles,

$$L = \sum_i \frac{m_i}{2} v_i^2$$

Note: Free particle L consistent with Galilean relativity

i.e.
$$\Delta L = \frac{m}{2}(\underline{v} + \underbrace{\underline{V}}_{\text{boost}})^2 - \frac{m v^2}{2} \overset{?}{=} \frac{df}{dt}$$

$$= \cancel{\frac{m v^2}{2}} + m \underline{v} \cdot \underline{V} + \frac{1}{2} m \underline{V}^2 - \cancel{\frac{m v^2}{2}}$$

$$= \frac{d}{dt}\left(m \underline{x} \cdot \underline{V} + \frac{1}{2} m V^2 t \right)$$

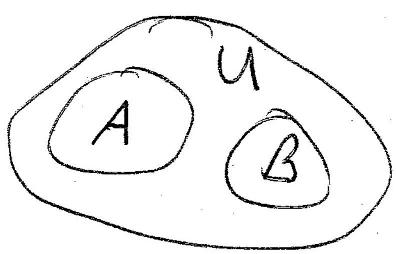
$\forall$

specifies f !

Now, useful to introduce concept of $\left\{ \begin{array}{l} \text{closed} \\ \text{open} \end{array} \right.$ system, i.e.

closed $\rightarrow$ non-interacting

open $\rightarrow$ interacting



Now, if U formed by two closed subsystems A, B, then

$L_u = L_A + L_B$ $\Rightarrow$ Lagrangians for closed sub-systems additive and can generalize to arbitrary # !

Now, for system of interacting particles (Galilean) which is closed, expect Lagrangian can be written as

$$L = \sum_i \frac{m_i}{2} v_i^2 + Q(\underline{r}_1, \underline{r}_2, \ldots)$$

interaction potential $\rightarrow$ function of coordinates

Now, in event that $q_i = \underline{x}_i$ (generalized coordinates are Cartesian coordinates) know Lagrange's eqns. must reduce to Newton's Laws

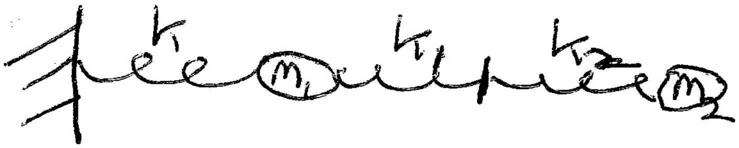
..E. $\Rightarrow$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}_i}\right) - \frac{\partial L}{\partial \underline{x}_i}$$
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}_i}\right) - \frac{\partial Q}{\partial \underline{x}_i}$$
$$= \frac{d}{dt}\left(\underline{p}_i\right) - \frac{\partial Q}{\partial \underline{x}_i}$$

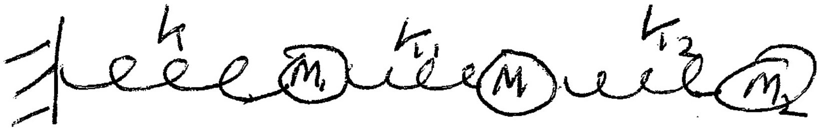
$\Rightarrow \quad Q = -U$ and

↳ potential energy

$$\boxed{L = T - U}$$

Examples; Theme: <u>Utility of Generalized Coordinates</u>

i) 

Observe: = 

with $M \to 0$

Coordinates: x_1, x, x_2

$$T = \frac{1}{2}\left(m_1 \dot{x}_1^2 + M\dot{x}^2 + m_2 \dot{x}_2^2\right)$$

For U, account for all "stored energy"!

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k_1 (x - x_1)^2 + \frac{1}{2} k_2 (x_2 - x)^2$$

$$L = T - U$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$m_1 \ddot{x}_1 + k x_1 - k_1 (X - x_1) = 0$$

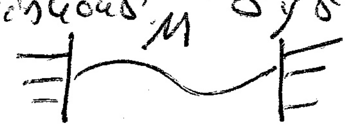
$$M \ddot{X} + k_1 (X - x_1) - k_2 (x_2 - X) = 0$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - X) = 0$$

$$\Rightarrow \begin{cases} (k_1 + k_2) X = k_1 x_1 + k_2 x_2 \\ \text{and reduce to 2 coupled oscillators.} \end{cases}$$

Continuum

iii) Derive NL equation for string with tension T (1D) – Lagrange Eqns. for Continuous System



$$S = \int dt \int dx \; \mathcal{L}$$

Lagrangian density

$$\mathcal{L} = \mathcal{L}\left(q(x,t), \dot{q}(x,t)\right)$$

Here: $q = y(x,t)$

$$T = \int_0^L dx\, \frac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2$$

$$U = \int_0^L ds\, T$$

$$ds^2 = dx^2 + dy^2$$
$$ds = (dx^2 + dy^2)^{1/2}$$

$$= \int_0^L dx \left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{1/2} T$$

$$S = \int dt \int dx \left\{ \frac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 - T\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{1/2} \right\}$$

$$= \int dt \int dx\, \mathcal{L}\left(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\right)$$

$$y_t = \frac{\partial y}{\partial t}$$
$$y_x = \frac{\partial y}{\partial x}$$

$$\delta S = \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial y_t}\delta y_t + \frac{\partial \mathcal{L}}{\partial y_x}\delta y_x \right\}$$

e.p.

$$= \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial y_t}\delta y\Big|_{t_1}^{t_2} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) + \frac{\partial \mathcal{L}}{\partial y_x}\delta y\Big|_0^L - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \right\}\delta y$$

e.p. space — fixed

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) = 0$$

Lagrange eqns.

$$\frac{\partial \mathcal{L}}{\partial y_t} = \mu y_t$$

$$\frac{\partial \mathcal{L}}{\partial y_x} = -T \frac{\partial y/\partial x}{\left(1+\left(\frac{\partial y}{\partial x}\right)^2\right)^{1/2}}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\left(\mu \frac{\partial y}{\partial t}\right) - \frac{\partial}{\partial x}\left(\frac{T\, \partial y/\partial x}{\left[1+(\partial y/\partial x)^2\right]^{1/2}}\right) = 0}$$

Linearizing, constant μ, T $\Rightarrow$

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}$$

Wave eqn.!

→ Miscellaneous Issues in Lagrangian Mechanics

i.) Recovering Newtonian Mechanics

(See 12a)

1.) Energy

→ isotropy of time for closed system

⇒ $\partial L/\partial t = 0$

⇒ energy $E = p\dot{q} - L$ conserved

i.e. $L = L(q, \dot{q}, t)$

$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \cancel{\frac{\partial L}{\partial t}}$, $\partial L/\partial \dot{q} = p$, $\frac{\partial L}{\partial q} = \dot{p}$

$dL/dt = \dot{p}\dot{q} + p\ddot{q} = \frac{d}{dt}(p\dot{q})$

$\therefore \frac{d}{dt}(p\dot{q} - L) = 0$

2.) Linear Momentum

Similarly

→ for closed system, origin of coordinate system is arbitrary, i.e. physics invariant upon $\underline{r} \to \underline{r} + \underline{\epsilon}$

$$\Rightarrow \delta L = \sum_i \frac{\partial L}{\partial \underline{r}_i} \cdot \underline{\epsilon} = \underline{\epsilon} \cdot \sum_i \frac{\partial L}{\partial \underline{r}_i}$$

$$\delta L = 0 \Rightarrow \sum_i \frac{\partial L}{\partial \underline{r}_i} = 0 \Rightarrow \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}_i}\right) = 0$$

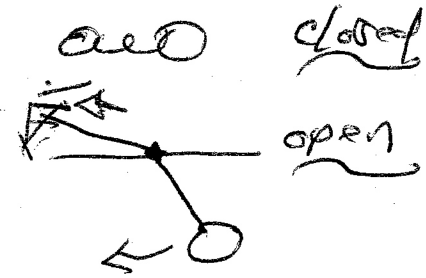
$$\Rightarrow \frac{d\underline{P}}{dt} = 0, \quad \underline{P} = \sum_i \partial L/\partial \underline{v}_i$$

Note: $$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

but $$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

and if $\partial L/\partial t = 0$ (no explicit time dependence) $\rightarrow$ {homogeneity of time for closed system

$\Rightarrow$
$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$



$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)$$

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0$$

$\therefore$ $$E = \dot{q}\frac{\partial L}{\partial \dot{q}} - L \quad \text{is constant}$$

$\downarrow$
energy

defines conservative system.

for $\partial L/\partial \underline{r}_i = -\partial U/\partial r_i \equiv \underline{F}_i$

$\Rightarrow \sum_i \underline{F}_i = 0 \quad \Rightarrow$ Newton's 3rd Law

gen. momentum

$$p_i \equiv \partial L/\partial \dot{q}_i \quad ; \quad \underbrace{F_i}_{\text{generalized force.}} = \partial L/\partial q_i$$

3a) Angular Momentum

→ isotropy of space (for closed system) ⇒ physics invariant under infinitesimal rotation

i.e. if $\delta \underline{\phi} \equiv$ vector infinitesimal rotation

$|\delta\phi| \rightarrow$ magnitude , $\dfrac{\delta\phi}{|\delta\phi|} \rightarrow$ axis of rotation.

then under rotation : $\delta \underline{r} = \delta\underline{\phi} \times \underline{r}$

$$\delta \underline{V} = \delta \underline{\phi} \times \underline{V}$$ (velocities change too)

⇒

Now, isotropy ⇒ $\delta L = 0$ upon rotation

$$\therefore \quad \delta L = \sum_i \Big(\underbrace{\frac{\partial L}{\partial \underline{r}_i}}_{\dot{\underline{p}}_i} \cdot \delta \underline{r}_i + \underbrace{\frac{\partial L}{\partial \underline{V}_i}}_{\underline{p}_i} \cdot \delta \underline{V}_i \Big) = 0$$

$$\Rightarrow \sum_i \left(\dot{\underline{p}}_i \cdot \delta\underline{\phi} \times \underline{r}_i + \underline{p}_i \cdot \delta\underline{\phi} \times \underline{v}_i \right) = \delta L = 0$$

re-arrange $\Rightarrow$

$$\delta\underline{\phi} \cdot \sum_i \left(\underline{r}_i \times \dot{\underline{p}}_i + \dot{\underline{r}}_i \times \underline{p}_i \right) = \delta\underline{\phi} \cdot \frac{d}{dt} \sum_i \underline{r}_i \times \underline{p}_i = 0$$

so $\frac{d}{dt} \underline{L} = 0$; $\boxed{\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i}$

angular momentum (conserved)

Note: Angular momentum depends on choice of origin, except when system at rest, as a whole.

i.e. $\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$

$$\underline{r} \rightarrow \underline{r}' + \underline{a}$$

$$\underline{L} = \sum_i \underline{r}_i' \times \underline{p}_i + \sum_i \underline{a} \times \underline{p}_i$$

$$= \underline{L}' + \underline{a} \times \underline{P}$$

Strongly recommend : Problem 3 ; Section 9 L+L.

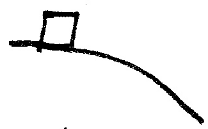
Constraints

i.e. $r_{ij} \to r_{ij}^2 = c_{ij}^2$

Ex ① rigid body (r_{ij} constant)

② gas in container (inside walls)

③ particle moving above spherical surface ($x^2+y^2+z^2-a^2 \geq 0$)



④ particle moving on wire

($x^2+y^2 = a^2$)

(30)

⑤ "rolling without slipping" $\underline{v} + \underline{r} \times \underline{\Omega} = 0$

pt contact stationary

Types:

i.) Holonomic: Expressible in form:

$$f(\underline{r}_1, \underline{r}_2 \ldots r_n, t) = 0$$

more generally: $f(q_1, q_2, \ldots q_n, t) = 0$

i.e. relation coordinates, time (only).

Ex ①, ④

ii.) Nonholonomic: all others.

i.e. inequalities (Ex. ③),

velocity dependence (Ex. ②), (Ex. ⑤ (albeit convertible))

Also: schleronomic $\longleftrightarrow$ independent time
rheonomic $\longleftrightarrow$ depends on time

Consider $\begin{cases} \text{holonomic constraint:} \\ \underline{\text{or}} \text{ nonholonomic of form } \left[df = \sum_k a_{\ell k}\, dq_k + a_{\ell t}\, dt \right] = 0 \end{cases}$

→ i.e. can eliminate directly!

$\alpha = 1 \ldots n; \quad f_\alpha(q_1, \ldots, t) = 0$

or retain, with Lagrange multiplier

For eqns. motion, extremize: → for force constr.

$$S' = \int dt \left(L(q_i; \dot{q}_i) + \sum_{\alpha=1}^{n} \lambda_\alpha f_\alpha(q_i) \right)$$

$\lambda_\alpha \longleftrightarrow$ Lagrange multiplier

$\begin{cases} m \text{ eqns.} \\ n \text{ constr.} \\ m+n \text{ Var} \end{cases}$ (n add'l → λ's)

∴ Lagrange Eqns:

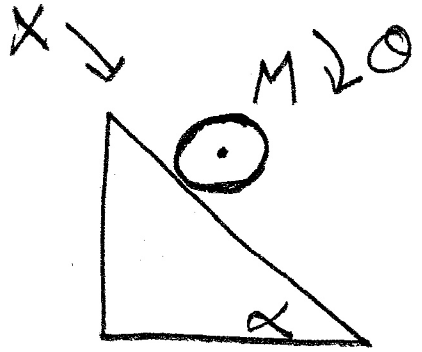
$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} - \sum_{\alpha=1}^{n} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} = 0}$$

Note: Can write $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} - \underbrace{Q_i^{con}}_{\text{gen. force}} = 0$

$$Q_i^{con} = \sum_{\alpha=1}^{n} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i}$$

Forces of constraint

Ex. 1



Cylinder rolling down incline
$(I = \frac{1}{2} M R^2)$

G.C.: x, θ

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$$

$$V = -Mgx \sin\alpha$$

Constraint: $x - R\theta = 0$

so

$$L = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 + Mg\sin\alpha \, x + \lambda (x - R\theta)$$

$\Rightarrow$

$$M\ddot{x} = Mg\sin\alpha + \lambda$$

force of constraint (i.e. friction $\Rightarrow$ roll without slip)

$$I\ddot{\theta} = -\lambda R$$

force of constraint.

$$x = R\theta \Rightarrow \ddot{x} = R\ddot{\theta}$$

$$\Rightarrow \lambda = \frac{-I\ddot{\theta}}{R} = \frac{-I\ddot{x}}{R^2}$$

and $\ddot{x} = \frac{2}{3} g \sin\alpha$ → acceleration due constraint

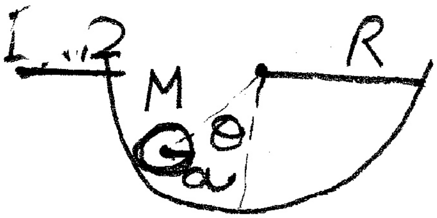
$\lambda = -\frac{1}{3} Mg \sin\alpha$ → force upward (friction → allows rolling)

Note : For holonomic constraint :

→ can eliminate directly, using $f(q_1, \ldots q_n, t) = 0$

* → but, using Lagrange multipliers allows determination of force of constraint.

impt



Sphere rolling cylinder
oscillation frequency, forces
of constraint ?

G.C. : θ, ϕ

$$T = \tfrac{1}{2} M (R-a)^2 \dot{\theta}^2 + \tfrac{1}{2} I \dot{\phi}^2$$

$$V = Mg(R-a)(1-\cos\theta)$$

$$(R-a)\theta - a\phi = 0 \quad \longleftrightarrow \text{ Eqn. Constraint (Rolling)}$$

so
$$L = \tfrac{1}{2} M (R-a)^2 \dot{\theta}^2 + \tfrac{1}{2} I \dot{\phi}^2 - Mg(R-a)(1-\cos\theta) + \lambda\big((R-a)\theta - a\phi\big)$$

$$\Rightarrow \quad \frac{d}{dt}\left(M(R-a)^2\dot{\theta}\right) = -Mg(R-a)\sin\theta + \lambda(R-a)$$

$$\frac{d}{dt}\left(I\dot{\phi}\right) = -\lambda a$$

λ → constraint 'force' (torque)

$$\therefore \begin{cases} (R-a)\ddot{\theta} = -g\sin\theta + \dfrac{\lambda}{M} \\ I\ddot{\phi} = -\lambda a \\ (R-a)\ddot{\theta} - a\ddot{\phi} = 0 \end{cases}$$

{3 eqn; 3 unknown

so $\quad \ddot{\phi} = \frac{(R-a)}{a} \ddot{\theta}$

$\Rightarrow \quad x = -\frac{I\,(R-a)}{Ma^2} \ddot{\theta}$

So
$$\left((R-a) + \frac{I(R-a)}{a^2} \right) \ddot{\theta} + g\,\theta = 0$$
for $\theta \ll 1$.

Note: Force of constraint varies in time

Aside: Rayleigh Dissipation Function

How include friction in Lagrangian mechanics (so as to get benefit of generalized coordinates)?

Observe: usually, simple friction has form:

$$\underline{F}_f = -k\underline{v}$$

so, can define:

$$\mathcal{F} = \frac{1}{2} k \underline{v}^2 = \frac{1}{2} k \dot{\underline{q}}^2$$

→ Rayleigh Disspn. Function (Power exerted vs. friction)

i.e.

$$\underline{F}_f = -\frac{\partial}{\partial \dot{\underline{q}}} \mathcal{F}$$

∴ can add a generalized force:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = 0}$$

generalized Lagrange Eqns.

{ incorporates friction in Lagrangian framework.

3.) Relation of Lagrangian Trajectories to Geodesics, for free particle.

Recall for free particle;

trajectory: $\delta S = 0$

$$S = \int dt\, T$$

$$T = \frac{1}{2} m \left(\frac{d\ell}{dt}\right)^2 \qquad \text{(minimizes action)}$$

for geodesic;

path: $\delta \int d\ell = 0$ (minimizes distance).

But

$$d\ell^2 = \sum_{i,k} g_{ik}(q)\, dq^i dq^k$$

$$d\ell = \left(\sum_{i,k} g_{ik}(q)\, dq^i dq^k\right)^{1/2}$$

Energy conserved $\Rightarrow$

$$E = \frac{1}{2} m \left(\frac{d\ell}{dt}\right)^2 = \frac{1}{2} m \sum_{i,k} g_{ik} \frac{dq^i}{dt} \frac{dq^k}{dt}$$

so $$dt = \left(\frac{m}{2E}\right)^{1/2} \left(\sum_{i,k} g_{ik}\, dq^i dq^k\right)^{1/2}$$

so

$$S = \int \frac{\frac{m}{2} \sum_{jk} g_{jk}\, dq^j dq^k}{\left(\frac{m}{2E}\right)^{1/2} \left(\sum_{jk} g_{jk}\, dq^j dq^k\right)^{1/2}}$$

$$= \left(\frac{Em}{2}\right)^{1/2} \int \left(\sum_{jk} g_{jk}\, dq^j dq^k\right)^{1/2}$$

$$= \left(\frac{Em}{2}\right)^{1/2} \int d\ell$$

$\therefore$ Action is simply distance (up to constant multiplier) for free particle.

$\Rightarrow$ Natural correspondence between free particle trajectories and geodesic curves.